Theoretical Aspects of Fiber Spinning

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ABSTRACT: A new coordinate expansion based on a Taylor series is used to derive the one-dimensional approximation to unsteady isothermal jet flows. The expansion procedure is carried out for an isothermal, Newtonian jet with no surface tension and air drag, and it can be used to derive a higher order approximation to the flow field. Two new formulations have been derived for the eigenvalue problem for fiber spinning stability for negligible inertia and gravity.
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INTRODUCTION

Many important problems in fluid mechanics involve the analysis of slender axisymmetric liquid jets. For example, in the manufacture of textile fibers, a fundamental process that must be analyzed is the continuous drawing of liquid filaments to form fibers. The spinning process can be roughly divided into three regions:[1,2] an extrusion or forming region close to the spinneret; a molten draw-down region in which a significant part of the fiber contraction occurs; and a cold draw-down or finishing region in which the material exhibits solid-like behavior. Most analyses of the spinning process focus on the molten draw-down region because of its processing importance and its mathematical tractability. Thus, as pointed out by Schultz and Davis,[3] these usual analyses of the continuous drawing of liquid fibers are valid in a region that is sufficiently far from the two ends of the fiber. The formulation of average boundary conditions at the beginning and end of the molten draw-down region avoids the need for analyses of the complicated flow fields in the extrusion and finishing regions.

For a complete analysis of the fluid mechanics in the molten draw-down region, the following theoretical aspects should be considered:

1. Formulation of appropriately simplified forms of the transport equations.
2. Solution of the steady-state forms of these equations.
3. Formulation and solution of the eigenvalue problem that describes the stability of the fiber to infinitesimal disturbances.
4. Finite-amplitude analysis of the transient response of the liquid fibers.

All four of the above aspects have been considered in a number of investigations.[1–14] In particular, much attention has been given to the problem of the isothermal spinning of an incompressible Newtonian fluid with dominant viscous effects and negligible inertia, gravity, surface tension, and air drag. It can be argued that viscous forces will often dominate the drawing process; however, the assumption of isothermal spinning of a Newtonian liquid is clearly only an approximate representation of a real industrial process. Nevertheless, a thorough analysis of the isothermal spinning of a Newtonian liquid with only viscous effects considered will permit one to assess the assumptions used to derive the simplified forms of the transport equations (by comparing theory and experiment) and also to gain some insight into the nature of the eigenvalue problem that results from the linear stability analysis. In addition, the information derived from the analysis of the simplified problem will provide a foundation for analyzing nonisothermal polymer jets with complex rheological behavior and with nonnegligible inertia, gravity, surface tension, and air drag effects.

Although the results for a Newtonian analysis have been derived by a number of investigators, further insights into the Newtonian problem (and hence to more complex fiber spinning problems) can be gained by considering somewhat different analyses of the problem, particularly in the derivation of the simplified forms of the transport equations and in the formulation of the eigenvalue problem for linear stability. The two objectives of this article are to formulate a systematic derivation of a one-dimensional unsteady

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approximation to the full axisymmetic fluid mechanics problem and to derive a simpler form for the eigenvalue problem for linear stability.

Schultz and Davis[3] have previously reviewed published derivations of the simplified one-dimensional equations generally used to analyze jet fluid mechanics. They concluded that all derivations[1,2,6] prior to their investigation were not completely acceptable either because a small perturbation parameter was not identified or because all of the relevant boundary conditions were not satisfied. Schultz and Davis implied that parametric expansions are better than coordinate expansions because the coordinate expansions that had been utilized could not satisfy conditions at all of the fiber boundaries. Schultz and Davis proposed a parametric expansion for a steady, axisymmetric jet by using a slenderness ratio as a perturbation parameter. A parametric expansion with the slenderness ratio as a perturbation parameter has also been used for unsteady jets.[13,14] Here, we carry out a systematic derivation of the simplied equations for an unsteady, axisymmetric jet by using a Taylor series for the axial velocity in the radial direction to formulate a coordinate expansion. The coordinate expansion analysis proposed here does lead systematically to the leading order equations (the so-called zero-order approximation)[12] for unsteady flow, does provide a method for calculating higher order corrections, and does satisfy all relevant boundary conditions to the appropriate order of approximation.

The eigenvalue problem for the linear stability of an isothermal Newtonian jet with dominant viscous effects has been considered by Pearson and Matovich,[4] Gelder,[5] Kase,[7] Fisher and Denn,[8] and Petrie.[11] Fisher and Denn solved a linear eigenvalue problem consisting of three ordinary differential equations using direct numerical integration, whereas Gelder used a finite-difference method to solve two ordinary differential equations. Petrie solved a two-equation eigenvalue problem essentially analytically, although the numerical solution of an algebraic equation was used to calculate the eigenvalues. In the investigations of Pearson and Matovich and of Kase, there is no direct study of the eigenvalues. Pearson and Matovich studied the effects of small perturbations in the boundary conditions, while Kase developed a transient solution that is a response to a step change in the tension in the fiber. In this article, we show that the eigenvalue problem for an isothermal Newtonian jet with dominant viscous forces can be representated by a single ordinary differential equation for the axial velocity. If the take-up velocity of the fiber is specified, the eigenvalue problem can be formulated either as a third-order ordinary differential equation (with two inlet boundary conditions and one exit boundary condition) or as a second-order ordinary differential equation (with an inlet Neumann condition and an integral boundary condition). An analytical solution can be derived for the eigenvalue problem, and the eigenvalues can be determined using tabulated functions. If the take-up force is specified, the jet dynamics are described by a second-order ordinary differential equation with two inlet boundary conditions. It is noted below that this initial value problem has only the trivial solution.

DERIVATION OF UNSTEADY ONE-DIMENSIONAL EQUATIONS

In this section, a systematic procedure is used to derive the unsteady, one-dimensional equations describing the fluid motion of an axisymmetric liquid jet. In the one-dimensional (or zero-order) approximation, both the axial velocity and the pressure will be independent of radial position in the jet. The proposed coordinate expansion procedure detailed here can be used to derive higher order approximations if required. We consider the unsteady, laminar flow of a slender, axisymmetric Newtonian jet of length L which empties into an inviscid gas phase. The liquid jet has constant density ρ and constant viscosity μ, the flow field is isothermal, the azimuthal velocity is zero, and there is no mass transfer to or from the gas phase. Gravity acts in the axial direction, the inlet axial velocity and jet radius are independent of time, and the gas phase pressure can be set equal to zero with no loss of generality. We also assume that both surface tension and air drag effects are negligible. Note that, if needed, surface tension effects could be included by using equations for a surface phase as boundary conditions at the gas–liquid interface rather than using the jump linear momentum equation. For this problem, the dimensionless continuity and Navier-Stokes equations can be written in cylindrical coordinates as follows:

$$\frac{\partial V}{\partial r} + \frac{V}{r} + \frac{\partial U}{\partial z} = 0 \tag{1}$$

$$\mathrm{Re}\left[\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial r} + U\frac{\partial V}{\partial z}\right] = -\frac{\partial p}{\partial r} + \left[\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \frac{V}{r^2} + \frac{\partial^2 V}{\partial z^2}\right] \tag{2}$$

$$\mathrm{Re}\left[\frac{\partial U}{\partial t} + V\frac{\partial U}{\partial r} + U\frac{\partial U}{\partial z}\right] = -\frac{\partial p}{\partial z} + \left[\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2}\right] + \frac{\mathrm{Re}}{\mathrm{Fr}} \tag{3}$$

Here,

$$\mathrm{Re} = \frac{\rho L U_a}{\mu} \tag{4}$$

$$\mathrm{Fr} = \frac{U_a^2}{gL} \tag{5}$$

where U_a is the average axial velocity at the inlet and g is the gravitational acceleration. Also, r and z are the dimensionless radial and axial coordinates, V and U are the dimensionless radial and axial velocities, t is the dimensionless time, and p is the dimensionless pressure. Dimensionless variables are defined in terms of their dimensional counterparts as follows:

$$r = \frac{r^*}{L}, \quad z = \frac{z^*}{L} \tag{6}$$

$$V = \frac{V^*}{U_a}, \quad U = \frac{U^*}{U_a} \tag{7}$$

$$t = \frac{t^* U_a}{L} \tag{8}$$

$$p = \frac{p^* L}{\mu U_a} \tag{9}$$

The boundary conditions at the gas–liquid interface can be formulated by utilizing appropriate jump balances. Because there is no mass transfer between phases, the jump mass balance reduces to the following form

$$V n_r + U n_z = u_n \tag{10}$$

where n_r and n_z are the radial and axial components of the unit normal vector to the surface pointing into the gas phase, V and U are the dimensionless liquid velocity components at the gas–liquid interface, and u_n is the normal component of the velocity of the moving surface. The gas–liquid interface is a time-dependent surface that can be represented by the function f in dimensionless form as follows:

$$f = r - R(z, t) = 0 \tag{11}$$

where the dimensionless jet radius R is defined as $R = R^*/L$. From eq. (11), the nonzero components of the unit normal vector are

$$n_r = \frac{1}{\left[1 + \left(\frac{\partial R}{\partial z}\right)^2\right]^{1/2}} \tag{12}$$

$$n_z = -\frac{\frac{\partial R}{\partial z}}{\left[1 + \left(\frac{\partial R}{\partial z}\right)^2\right]^{1/2}} \tag{13}$$

and also

$$\frac{\partial f}{\partial t} = -\frac{\partial R}{\partial t} \tag{14}$$

Because the normal surface velocity can be represented by the equation[15]

$$u_n = -\frac{\frac{\partial f}{\partial t}}{[\nabla f \cdot \nabla f]^{1/2}} \tag{15}$$

introduction of eqs. (12)–(15) into eq. (10) produces the final result for the jump mass balance:

$$V\,(r = R) - U\,(r = R)\left[\frac{\partial R}{\partial z}\right] = \frac{\partial R}{\partial t} \tag{16}$$

Because there is no stress in the gas phase, and because there is no mass transfer between phases, the jump linear momentum equation reduces to

$$\boldsymbol{T} \cdot \boldsymbol{n} = \boldsymbol{0} \tag{17}$$

where the dimensionless stress $\boldsymbol{T}$ in the liquid phase is defined as $\boldsymbol{T} = L\boldsymbol{T}^*/\mu U_a$. Introduction of eqs. (12) and (13) and utilization of the constitutive equation for an incompressible Newtonian fluid yield the following equations for the r and z components of the jump linear momentum equation, valid at $r = R$:

$$p = 2\frac{\partial V}{\partial r} - \frac{\partial R}{\partial z}\left(\frac{\partial V}{\partial z} + \frac{\partial U}{\partial r}\right) \tag{18}$$

$$\frac{\partial V}{\partial z} + \frac{\partial U}{\partial r} = \frac{\partial R}{\partial z}\left[-p + 2\frac{\partial U}{\partial z}\right] \tag{19}$$

The θ component is satisfied identically.

The problem formulation is completed by writing down appropriate conditions on the jet axis and on the inlet and outlet surfaces. Along the jet axis, continuous behavior in the transport equations requires that

$$V = 0, \quad r = 0 \tag{20}$$

$$\frac{\partial U}{\partial r} = 0, \quad r = 0 \tag{21}$$

At the inlet surface to the jet, it is assumed that there is a uniform axial velocity equal to the cross-sectional average so that

$$U = 1, \quad z = 0 \tag{22}$$

At the outlet surface, if the takeup velocity is specified, the appropriate end condition is

$$U = D_R, \quad z = 1 \tag{23}$$

where D_R is the ratio of outlet to inlet axial velocities. If the takeup force is specified, then the tension force is the same everywhere in a Newtonian jet when inertia, gravity, and surface tension contributions can be ignored.

The above set of equations is exact but difficult to solve. For slender, axisymmetric jets, we seek a zero-order solution by writing a Taylor series around $r = 0$ for the axial velocity at any axial position z:

$$U(r, z, t) = U(0, z, t) + \left(\frac{\partial U}{\partial r}\right)_{r=0} r + \left(\frac{\partial^2 U}{\partial r^2}\right)_{r=0} \frac{r^2}{2} + \cdots \tag{24}$$

If eq. (21) is introduced, this result can be rewritten as

$$U(r, z, t) = U_0(z, t) + U_1(z, t)\frac{r^2}{2} + O(r^3) \tag{25}$$

and this equation serves to define U_0 and U_1. This Taylor series will provide a good representation for the axial velocity for small values of r. For a slender jet, r should be small across the entire jet radius so that a one-term Taylor series should provide an acceptable approximation to U. Substitution of eq. (25) into the continuity equation, eq. (1), produces the following result for V when the equation is integrated and eq. (20) is utilized:

$$V = -\left(\frac{\partial U_0}{\partial z}\right)\frac{r}{2} - \left(\frac{\partial U_1}{\partial z}\right)\frac{r^3}{8} + O(r^4) \tag{26}$$

Also, substitution of eqs. (25) and (26) into the radial component of the Navier-Stokes equation, eq. (2), yields upon integration:

$$p = p(0, z, t) + O(r^2) = p_0(z, t) + O(r^2) \tag{27}$$

Finally, substitution of eqs. (25)–(27) into the axial component of the Navier-Stokes equation, eq. (3), leads to the following equation:

$$\mathrm{Re}\left[\frac{\partial U_0}{\partial t} + U_0\frac{\partial U_0}{\partial z}\right] = -\frac{\partial p_0}{\partial z} + 2U_1 + \frac{\partial^2 U_0}{\partial z^2} + \frac{\mathrm{Re}}{\mathrm{Fr}} + O(r) \tag{28}$$

Equation (28) is the basic equation that can be used for the determination of U_0 if p_0 and U_1 can be eliminated. This can be accomplished by using the radial and axial components of the jump linear momentum equation.

Introduction of eqs. (25)–(27) into eqs. (18) and (19) yields the following modified forms of the two components of the jump linear momentum equation after some rearrangements:

$$p_0 = -\frac{\partial U_0}{\partial z} + O(R^2) \tag{29}$$

$$U_1 R - \left(\frac{\partial^2 U_0}{\partial z^2}\right)\frac{R}{2} = 3\frac{\partial R}{\partial z}\frac{\partial U_0}{\partial z} + O(R^2) \tag{30}$$

Substitution of these two results into eq. (28) produces the following form of the axial component of the Navier-Stokes equation:

$$\mathrm{Re}\left[\frac{\partial U_0}{\partial t} + U_0\frac{\partial U_0}{\partial z}\right] = \frac{\mathrm{Re}}{\mathrm{Fr}} + 3\frac{\partial^2 U_0}{\partial z^2} + \frac{6}{R_0}\frac{\partial R_0}{\partial z}\frac{\partial U_0}{\partial z} \tag{31}$$

The dimensionless jet radius R is written as R_0 because, as indicated below, it is based on using U_0 for the axial velocity. Substituting eqs. (25) and (26) into the exact form of the jump mass balance, eq. (16), gives, after some rearrangement, results which illustrate that R_0 is directly related to U_0 when higher order terms are neglected:

$$\frac{\partial A_0}{\partial t} + \frac{\partial (A_0 U_0)}{\partial z} = 0 \tag{32}$$

$$A_0 = R_0^2 \tag{33}$$

Equations (31) and (32) thus describe the unsteady dynamics of the isothermal Newtonian jet. These equations are equivalent to the unsteady-state equations proposed by Geyling[2] when both gravity and surface tension are excluded. However, Geyling derived his results by simply assuming that the axial velocity and pressure were independent of r and by integrating the transport equations over the fiber cross section, and he did not provide the details of his derivation.

At steady state, for the special use of $\mathrm{Re} = 0$ and $\mathrm{Fr}^{-1} = 0$, the velocity field is given by the following equations if the takeup velocity is specified:

$$U_0 = e^{\alpha z} \tag{34}$$

$$\alpha = \ln D_R \tag{35}$$

For the same special case, the velocity profile takes an equivalent form if the takeup force is specified:

$$U_0 = e^{\beta z} \tag{36}$$

Here, however, β is a constant related to the takeup force.

The above derivation is based on using a coordinate expansion in terms of a Taylor series for the axial velocity. The success of the approximation depends on having a slender jet so that the dimensionless radial coordinate r is small.

FORMULATION OF EIGENVALUE PROBLEM

A linearized stability analysis for this jet problem can be carried out in the usual manner by considering solutions close to the steady solutions and by formulating an eigenvalue problem that can be solved to determine the fate of infinitesimal perturbations from the steady flow. We consider the following forms of eqs. (31) and (32)

$$\mathrm{Re}\left[\frac{\partial \ln U_0}{\partial t} + \frac{\partial U_0}{\partial z}\right] = \frac{\mathrm{Re}}{\mathrm{Fr}\, U_0} + 3\,\frac{\partial^2 \ln U_0}{\partial z^2} + 3\left[\frac{\partial \ln U_0}{\partial z} + \frac{\partial \ln A_0}{\partial z}\right]\frac{\partial \ln U_0}{\partial z} \tag{37}$$

$$\frac{1}{U_0}\frac{\partial \ln A_0}{\partial t} + \frac{\partial \ln U_0}{\partial z} + \frac{\partial \ln A_0}{\partial z} = 0 \tag{38}$$

and propose solutions of the form

$$U_0(z,\,t) = U_s(z)[1 + \hat{U}(z)e^{qt}] \tag{39}$$

$$\ln U_0 = \ln U_s + \hat{U}e^{qt} \tag{40}$$

$$A_0(z,\,t) = A_s(z)[1 + \hat{A}(z)e^{qt}] \tag{41}$$

$$\ln A_0 = \ln A_s + \hat{A}e^{qt} \tag{42}$$

where U_s and A_s are the appropriate steady solutions and $\hat{U}$ and $\hat{A}$ represent small perturbations to the steady flow. Note that we are studying the behavior of a typical mode of disturbance by considering only a single component of a series representation. Substitution of eqs. (39)–(42) into eqs. (37) and (38) produces the following equations for the perturbed variables:

$$\mathrm{Re}\left[q\hat{U} + \hat{U}\frac{dU_s}{dz} + U_s\frac{d\hat{U}}{dz}\right] = -\frac{\mathrm{Re}\,\hat{U}}{\mathrm{Fr}\,U_s} + 3\left[\frac{d^2\hat{U}}{dz^2} + \frac{d\ln U_s}{dz}\left(\frac{d\hat{U}}{dz} + \frac{d\hat{A}}{dz}\right)\right] \tag{43}$$

$$\hat{A}q + U_s\left(\frac{d\hat{U}}{dz} + \frac{d\hat{A}}{dz}\right) = 0 \tag{44}$$

For the special case of Re = 0, $\mathrm{Fr}^{-1} = 0$, and specified takeup velocity, eqs. (43) and (44) can be combined with eq. (34) to give the following results:

$$\frac{d^2\hat{U}}{dz^2} + \alpha\left(\frac{d\hat{U}}{dz} + \frac{d\hat{A}}{dz}\right) = 0 \tag{45}$$

$$\hat{A}q + e^{\alpha z}\left(\frac{d\hat{U}}{dz} + \frac{d\hat{A}}{dz}\right) = 0 \tag{46}$$

Substitution of eq. (46) into eq. (45) gives

$$e^{\alpha z}\frac{d^2\hat{U}}{dz^2} - \alpha\hat{A}q = 0 \tag{47}$$

Differentiation of eq. (47) and substitution of eq. (45) yield the following third-order ordinary differential equation:

$$\frac{d}{dz}\left[e^{\alpha z}\frac{d^2\hat{U}}{dz^2}\right] + q\,\frac{d^2\hat{U}}{dz^2} + \alpha q\,\frac{d\hat{U}}{dz} = 0 \tag{48}$$

Two of the boundary conditions for this equation follow from eqs. (22), (23), (34), and (39):

$$\hat{U}(0) = 0 \tag{49}$$

$$\hat{U}(1) = 0 \tag{50}$$

In addition, because the inlet radius to the jet is independent of time:

$$\hat{A}(0) = 0 \tag{51}$$

Thus, the combination of eqs. (47) and (51) provides a third boundary condition:

$$\frac{d^2\hat{U}}{dz^2}(0) = 0 \tag{52}$$

Equations (48)–(50) and (52) represent a third-order eigenvalue problem with eigenvalue q and $\alpha > 0$. The operator generated by the third-order differential expression and the three boundary conditions is not self-adjoint. Properties of third-order linear differential equations are discussed by Gregus.[16]

A different formulation of the same eigenvalue problem can be derived by defining a new variable W:

$$W = \frac{d\hat{U}}{dz} \tag{53}$$

Utilization of this definition in eqs. (48) and (52) leads to a second-order differential equation

$$\frac{d}{dz}\left[e^{\alpha z}\frac{dW}{dz}\right] + q\frac{dW}{dz} + \alpha qW = 0 \tag{54}$$

and to a Neumann inlet condition:

$$\frac{dW}{dz}(0) = 0 \tag{55}$$

In addition, integration of eq. (53) and application of eqs. (49) and (50) produces an integral boundary condition:

$$\int_0^1 Wdz = 0 \tag{56}$$

Again, the operator generated by the second-order differential expression and the two boundary conditions is not self-adjoint. Note that in both formulations of the eigenvalue problem, only a single differential equation is involved, and this is a new result which facilitates either numerical or analytical solution of the eigenvalue problem. The second formulation, with the integral boundary condition, is used to obtain an analytical solution to the problem in the next section. Krall[17,18] has discussed various aspects of eigenvalue problems with integral boundary conditions.

In general, the eigenvalue q can be represented as follows:

$$q = q_R + iq_I \tag{57}$$

For the marginal or neutral state, $q_R = 0$, and there is a stationary marginal state if $q_I = 0$ (principle of exchange of stabilities) or an oscillatory marginal state if $q_I \neq 0$ (overstability). Because both versions of the above eigenvalue problem involve differential operators that are not self-adjoint, it is difficult to characterize the nature of the eigenvalues before actually solving the eigenvalue problem. However, it is possible to determine whether the marginal or neutral state is stationary by setting $q = 0$ in eq. (48):

$$\frac{d}{dz}\left[e^{\alpha z}\frac{d^2\hat{U}}{dz^2}\right] = 0 \tag{58}$$

For any value of α, the only solution of eq. (58) subject to eqs. (49), (50), and (52) is

$$\hat{U} = 0 \tag{59}$$

so that the marginal state cannot be stationary.

A second problem of interest is one for which Re $= 0$, $\mathrm{Fr}^{-1} = 0$, and the takeup force is specified. For this case, it is easy to show [using eq. (36) and the fact that the takeup force does not change with time] that one equation for the perturbed variables is the following:

$$\frac{d\hat{U}}{dz} + \beta[\hat{U} + \hat{A}] = 0 \tag{60}$$

A second equation for the perturbed variables can be obtained by combining eqs. (36) and (44):

$$\hat{A}q + e^{\beta z}\left[\frac{d\hat{U}}{dz} + \frac{d\hat{A}}{dz}\right] = 0 \tag{61}$$

Differentiation of eq. (60) gives

$$\frac{1}{\beta}\frac{d^2\hat{U}}{dz^2} + \frac{d\hat{U}}{dz} + \frac{d\hat{A}}{dz} = 0 \tag{62}$$

and substitution of eqs. (60) and (62) into eq. (61) produces the following second-order differential equation for $\hat{U}$:

$$e^{\beta z}\frac{d^2\hat{U}}{dz^2} + q\frac{d\hat{U}}{dz} + q\beta\hat{U} = 0 \tag{63}$$

As before, eqs. (22), (36), and (39) produce a Dirichlet initial condition

$$\hat{U}(0) = 0 \tag{64}$$

and, furthermore, eqs. (51), (60), and (64) yield a Neumann initial condition:

$$\frac{d\hat{U}}{dz}(0) = 0 \tag{65}$$

The initial value problem given by eqs. (63)–(65) has only the solution

$$\hat{U} = 0 \tag{66}$$

because the only solution of a completely homogeneous initial value problem is the trivial solution.[19] Consequently, the fiber spinning problem with takeup force prescribed is always stable. A similar conclusion was reached by Pearson and Matovich[4] using an entirely different method.

SOLUTION OF EIGENVALUE PROBLEM

The two formulations of the linear eigenvalue problem for $\mathrm{Re} = 0$, $\mathrm{Fr}^{-1} = 0$, and specified takeup velocity both involve a single ordinary differential equation for the perturbed velocity variable. As noted above, the fact that the eigenvalue problem has only one differential equation facilitates either numerical or analytical solution of the problem. We now show how the second formulation of the eigenvalue problem can be used to generate an analytical solution. Two integrations of eq. (54) from $z = 0$ to $z = z$ and introduction of eq. (55) give the result

$$W + \alpha \int_0^z W dz' - \alpha \exp\left(\frac{q}{\alpha} e^{-\alpha z}\right) \times \int_0^z W \exp\left(-\frac{q}{\alpha} e^{-\alpha z'}\right) dz' = K \quad (67)$$

where K is a constant. Equation (67) is a linear, nonhomogeneous Volterra integral equation of the second kind for W with solution

$$W = K \exp\left(-\frac{q}{\alpha}\right) \times \left\{\exp\left(\frac{q}{\alpha} e^{-\alpha z}\right) + q e^{-\alpha z} \int_0^z \exp\left[\frac{q}{a} e^{-\alpha z'}\right] dz'\right\} \quad (68)$$

The integral boundary condition, eq. (56), requires that the following equation be satisfied:

$$\left(1 - \frac{q}{\alpha} e^{-\alpha}\right) \int_0^1 \exp\left(\frac{q}{\alpha} e^{-\alpha z}\right) dz - \frac{\exp\left(\frac{q}{\alpha} e^{-\alpha}\right)}{\alpha} + \frac{\exp\left(\frac{q}{\alpha}\right)}{\alpha} = 0 \quad (69)$$

Because $q = iq_I$ at neutral stability, this equation leads to the following two equations that can be used to determine α and q_I:

$$-\int_b^{be^{-\alpha}} \frac{\cos k}{k} dk - e^{-\alpha} b \int_b^{be^{-\alpha}} \frac{\sin k}{k} dk - \cos(be^{-\alpha}) + \cos b = 0 \quad (70)$$

$$-\int_b^{be^{-\alpha}} \frac{\sin k}{k} dk + e^{-\alpha} b \int_b^{be^{-\alpha}} \frac{\cos k}{k} dk - \sin(be^{-\alpha}) + \sin b = 0 \quad (71)$$

$$b = \frac{q_I}{\alpha} \quad (72)$$

Equations (70) and (71) can be solved by an iterative approach using tables for the sine and cosine integrals.[20] The values of α and q_I determined from these equations are in good agreement with values of $\alpha = 3.006$ and $q_I = 13.989$ calculated by Fisher and Denn[8] by direct numerical integration. Finally, the perturbed velocity $\hat{U}$ can be determined by direct integration using eqs. (49), (53), and (68):

$$\hat{U} = K \exp\left(-\frac{q}{\alpha}\right)\left[\left(1 - \frac{qe^{-\alpha z}}{\alpha}\right) \int_0^z \exp\left(\frac{q}{\alpha} e^{-\alpha z'}\right) dz' - \frac{\exp\left(\frac{q}{\alpha} e^{-\alpha z}\right)}{\alpha} + \frac{\exp\left(\frac{q}{\alpha}\right)}{\alpha}\right] \quad (73)$$

Equations (69) and (73) are similar in form to results presented by Petrie.[11]

CONCLUDING REMARKS

In this article, we have proposed using a coordinate expansion based on a Taylor series to derive the one-dimensional approximation to unsteady isothermal jet flows. This coordinate expansion does satisfy all relevant boundary conditions, an objection to previous coordinate expansions. The proposed method can be used to derive higher order approximations to the flow field. This can be done if the calculated U_0 and U_1 are used in a higher order version of eq. (32). As noted above, the derived leading order equations are not new since parametric perturbation methods have been previously used to derive both the steady[3] and unsteady[13,14] equations. However, the application of a coordinate expansion rather than a parametric expansion provides a different method of analyzing unsteady jet flows that may be advantageous in some cases.

We have also proposed two new formulations of the eigenvalue problem for an isothermal, Newtonian jet with $\mathrm{Re} = 0$, $\mathrm{Fr}^{-1} = 0$, negligible surface tension and air drag, and specified takeup velocity. This formulation involves only a single differential equation, facilitating either numerical or analytical solution of the eigenvalue problem. We have also derived a new formulation for the case of $Re = 0$, $Fr^{-1} = 0$, and specified takeup force, and also have shown that the flow is always stable.

The coordinate expansion procedure was carried out for an isothermal, Newtonian jet with no surface tension and air drag, and the eigenvalue problem was limited to this case with the further assumptions of $Re = 0$, Fr^{-1} 0. However, the techniques developed here could be useful when a nonisothermal, viscoelastic jet is analyzed and/or when inertia, gravity, surface tension, and air drag effects are not excluded.

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